HEAT CONDUCTION AND THERMAL ELASTICITY EQUATIONS FOR OPERATING ELEMENTS OF METAL CERAMIC CASES (MCC) CONTAINING SECTOR- AND WEDGE-SHAPED FOREIGN INCLUSIONS

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Partly degenerate differential heat conduction and thermal elasticity equations with coefficients of the generalized function type are derived for investigation of thermal processes in inhomogeneous operating MCC elements containing sector- and wedge-shaped foreign through inclusions during their manufacture and operation. The process of finding solutions for the derived equations is illustrated by an example of the heat conduction problem for a plate with a wedge-shaped inclusion having the opening angle $2\varphi_0$.

The development of metal ceramic cases for integrated circuits (IC) is rapidly approaching the stage when it will become nearly as important as the development of IC themselves. A crystal IC with the highest speed and closest packing that can be invented becomes useless if it cannot be connected electrically, mechanically, and thermally with the subsequent level of radio electronic complex compounds. Therefore, MCC developers and users are always faced with a problem whose complexity increases from day to day [1, 2]. During MCC manufacture and operation their metal ceramic connections experience substantial thermal effects. The serviceability of such inhomogeneous metal ceramic elements at high temperatures depends on their geometrical shape, the physical and mechanical properties of the materials and the operating conditions. It should be noted that whereas the operating conditions are usually prescribed, the first two factors can vary and should be considered in close relation with each other. One of the crucial criteria for choosing materials for MCC and their design is material thermal strength by which is meant here the ability [3-5] of the operating metal ceramic elements to resist heat fluxes without any destruction.

It was found [6-8] that the wear resistance and types of destruction of solid alloys depend on the properties and structural characteristics of particular components.

Studies of temperature stresses in ceramic objects, which are necessary for estimating their thermal strength, should be conducted with their characteristic features exhibited in experiments, primarily, mechanisms of ceramic deformation associated with the material structure taken into consideration [9]. The strain diagram field for dense oxide ceramics at temperatures of $0.5T_{melt}$ is practically linear up to destruction [10]. Therefore studies of temperature stresses and strains in dense ceramic bodies over a temperature range from 300 K to $0.5T_{melt}$ can be carried out within the framework of elasticity theory.

As a result of differences in the temperature coefficients of linear expansion (TCLE) of the matrix and the foreign inclusion materials at their connection, temperature stresses which arise with a great difference of TCLE may be very large [7]. Since the strength properties of the matrix materials may be quite low, for example, for particular types of ceramics, a foreign inclusion can lead to destruction of the connection. Therefore the problem of finding the thermally stressed state in the metal-ceramic connection region is very important [11-13].

In what follows the operating elements of MCC containing sector- and wedge-shaped inclusions will be assumed to be thin-walled piecewise homogeneous structures consisting of individual parts with different physical and mechanical properties which are constant within each of the parts. In each specific case the physical and mechanical properties of a piecewise homogeneous body as an entity may be described in terms of asymmetric unit functions [14]. The apparatus of generalized functions appears to be effective for derivation of the basic heat conduction and thermal elasticity equations and their subsequent solution [15].

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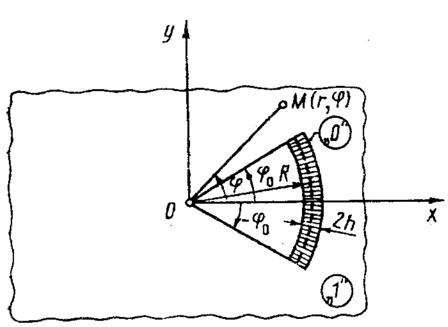

Fig. 1

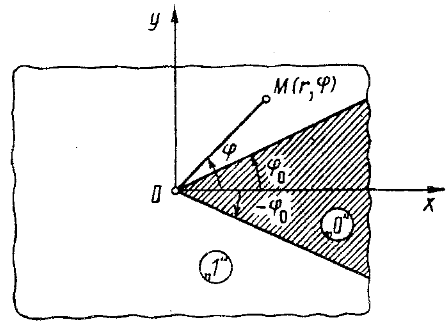

Fig. 2

Fig. 1. A plate with a foreign inclusion shaped like an annular sector with the median radius R (m) and wall thickness 2h (m).

Fig. 2. A plate containing a foreign through wedge-shaped inclusion with the opening angle $2\varphi_0$ (rad).

An inhomogeneous operating MCC element which represents the metal ceramic body studied will be simulated by a thin plate, 2δ in thickness, containing a thin-walled through foreign inclusion formed like an annular sector with the median radius R and wall thickness 2h (Fig. 1).

Heat transfer between the plate surfaces $z = \pm\delta$ and the environment follows Newton's law, the temperatures of fluids flowing over the surfaces being equal ($t^{+nf} = t^{-}_{f} = t_f$). For the description of heat conduction in a inhomogeneous plate use will be made of a system of simultaneous differential equations for the integral characteristics T, T^* of the temperature $t(r, \varphi, z, \tau)$ [16, 17]:

$$\begin{gathered}
\Lambda(r,\varphi)\Delta T + \Lambda^*(r,\varphi)\Delta T^* + \frac{\partial\Lambda(r,\varphi)}{\partial r}\frac{\partial T}{\partial r} + \frac{\partial\Lambda^*(r,\varphi)}{\partial r}\frac{\partial T^*}{\partial r} + \\
+ \frac{\partial\Lambda(r,\varphi)}{d\varphi}\frac{1}{r^2}\frac{\partial T}{\partial\varphi} + \frac{\partial\Lambda^*(r,\varphi)}{\partial\varphi}\frac{1}{r^2}\frac{\partial T^*}{\partial\varphi} - \alpha_+(r,\varphi)[T - \\
- t^c_+(r,\varphi,\tau)] - \alpha_-(r,\varphi)[T^* - t^c_-(r,\varphi,\tau)] = C(r,\varphi)\dot{T} + C^*(r,\varphi)\dot{T}^*, \\
\Lambda^{**}(r,\varphi)\Delta T^* + 3\Lambda^*(r,\varphi)\Delta T + 3\frac{\partial\Lambda^*(r,\varphi)}{\partial r}\frac{\partial T}{\partial r} + \\
+ \frac{\partial\Lambda^{**}(r,\varphi)}{\partial r}\frac{\partial T^*}{\partial r} + 3\frac{\partial\Lambda^*(r,\varphi)}{\partial\varphi}\frac{1}{r^2}\frac{\partial T}{\partial\varphi} + \frac{\partial\Lambda^{**}(r,\varphi)}{\partial\varphi}\times \\
\times\frac{1}{r^2}\frac{\partial T^*}{\partial\varphi} - 3\left\{\left[\alpha_+(r,\varphi) + \frac{1}{\delta^2}\Lambda(r,\varphi)\right]T^* - \alpha_+(r,\varphi)t^c_-(r,\varphi,\tau) + \right. \\
\left. + \alpha_-(r,\varphi)[T - t^c_+(r,\varphi,\tau)]\right\} = 3C^*(r,\varphi)T + C^{**}T^*.
\end{gathered} \tag{1}$$

with the definitions

$$\begin{gathered}
\Lambda(r,\varphi) = \int_{-\delta}^{\delta}\lambda_t(r,\varphi,z)\,dz;\quad \Lambda^*(r,\varphi) = \frac{1}{\delta}\int_{-\delta}^{\delta} z\lambda_t(r,\varphi,z)\,dz; \\
C(r,\varphi) = \int_{-\delta}^{\delta} c_v(r,\varphi,z)\,dz;\quad C^*(r,\varphi) = \frac{1}{\delta}\int_{-\delta}^{\delta} zc_v(r,\varphi,z)\,dz; \\
\Lambda^{**}(r,\varphi) = \frac{3}{\delta^2}\int_{-\delta}^{\delta} z^2\lambda_t(r,\varphi,z)\,dz;\quad C^{**}(r,\varphi) = \frac{3}{\delta^2}\int_{-\delta}^{\delta} z^2c_v(r,\varphi,z)\,dz; \\
\alpha_\pm = \alpha_z^+ \pm \alpha_z^-;\quad t^c_\pm = \frac{t_c^+ \pm t_c^-}{2};\quad T_{\pm} = \frac{1}{2\delta}\int_{-\delta}^{\delta} t^*(r,\varphi,z,\tau)\,dz;
\end{gathered} \tag{2}$$

$T^* = \frac{3}{2\delta^2}\int_{-\delta}^{\delta} zt^*(r, \varphi, z, \tau)dz$; $t^*(r, \varphi, z, \tau) = t(r, \varphi, z, \tau) - t_0$ — is the body temperature increment; $c_v(r, \varphi, z) = c(r, \varphi, z)\rho(r, \varphi, z)$ is the volumetric heat capacity of the inhomogeneous plate; $\Delta = \partial^2/\partial r^2 + 1/r\cdot\partial/\partial r + 1/r^2\cdot\partial^2/\partial\varphi^2$; t^+_c, t^-_c are the temperatures of fluids flowing over the upper $z = +\delta$ and lower $z = -\delta$ plate surface; $T = \partial T/\partial r$.

The physical and mechanical properties of a plate with an inclusion of annular sector type will be expressed as [17]

$$p(r, \varphi) = p_1 + 2h(p_0 - p_1)\delta(r - R)N(\varphi),$$
$$N(\varphi) = S_-(\varphi + \varphi_0) - S_+(\varphi - \varphi_0),\ \delta(r - R) = \lim_{h\to 0}\frac{S_-(r - R + h) - S_+(r - R - h)}{2h} \quad (3)$$

Then, for the thermal elasticity problem symmetric relative to the median plane of the plate we have $T^* = 0$ and from relations (2) we can write [17]

$$\Lambda(r, \varphi) = 2\delta[\lambda_t^{(1)} + 2h(\lambda_t^{(0)} - \lambda_t^{(1)})\delta(r - R)N(\varphi)],\ \Lambda^*(r, \varphi) = 0,$$
$$C(r, \varphi) = 2\delta[c_v^{(1)} + 2h(c_v^{(0)} - c_v^{(1)})\delta(r - R)N(\varphi)],\ C^*(r, \varphi) = 0. \quad (4)$$

After substitution of expressions (4) and some transformations, the system of simultaneous heat conduction equations (1) as applied to the process of firing the MCC elements considered, which ensures their symmetrical heating, can be reduced to one heat conduction equation for T:

$$\Delta T - \varkappa_1^2(T - t_c) = \frac{1}{a_1}\dot{T} + 2h(1 - K_\lambda)\left.\frac{\partial T}{\partial r}\right|_{r=R} N(\varphi)\delta'(r - R) +$$
$$+ 2h\left\langle (1 - K_\lambda)\frac{1}{R}\left\{\left(\left.\frac{\partial T}{\partial r}\right|_{r=R} + \frac{1}{R}\left.\frac{\partial^2 T}{\partial\varphi^2}\right|_{r=R}\right)N(\varphi) +\right.\right.$$
$$+ \frac{1}{R}\left[\left.\frac{\partial T}{\partial\varphi}\right|_{\substack{r=R\\ \varphi=-\varphi_0+0}}\delta_-(\varphi + \varphi_0) - \left.\frac{\partial T}{\partial\varphi}\right|_{\substack{r=R\\ \varphi=\varphi_0-0}}\delta_+(\varphi - \varphi_0)\right]\Bigg\} + \quad (5)$$
$$+ \left[\left(\frac{c_v^{(0)}}{\lambda_t^{(1)}} - \frac{1}{a_1}\right)\dot{T}|_{r=R} + \left(\frac{\alpha_0}{\lambda_t^{(1)}\delta} - \varkappa_1^2\right)(T - t_c)|_{r=R}\right]N(\varphi)\Big\rangle\delta(r - R),$$

where α_0 and α_1 are the coefficients of heat transfer from the inclusion and the basic material surfaces: $a_1 = \lambda^{(1)}_t/c^{(1)}_v$ is the thermal diffusivity of the basic material; $\delta_\pm(\xi) = dS_\pm(\xi)/d\xi$; $\delta'(\xi) = d\delta(\xi)/d\xi$; $\varkappa^2_i = \alpha_i/\lambda^{(1)}_t\delta$; $K_\lambda = \lambda^{(0)}_t/\lambda^{(1)}_t$; $i = 0.1$.

The expressions for the coefficients

$$J_1 = \int_{-\delta}^{\delta}\frac{\lambda\mu}{\lambda + 2\mu}dz,\ J_2 = \int_{-\delta}^{\delta}\mu dz,\ J_3 = \int_{-\delta}^{\delta}\frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\alpha_t t^*(r, \varphi, z, \tau)dz,$$
$$J_4 = \int_{-\delta}^{\delta} z\frac{\mu(\lambda + \mu)}{\lambda + 2\mu}dz,\ J_5 = \int_{-\delta}^{\delta} z^2\frac{\lambda\mu}{\lambda + 2\mu}dz,\ J_1^* = \int_{-\delta}^{\delta} z\frac{\lambda\mu}{\lambda + 2\mu}dz,$$
$$J_2^* = \int_{-\delta}^{\delta} z\mu dz,\ J_3^* = \int_{-\delta}^{\delta} z\frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\alpha_t t^*(r, \varphi, z, \tau)dz, \quad (6)$$
$$J_4^* = \int_{-\delta}^{\delta} z^2\frac{\mu(\lambda + \mu)}{\lambda + 2\mu}dz,\ J_2^{(1)} = \int_{-\delta}^{\delta} z^2\mu dz$$

are the definite integrals entering into the system of simultaneous thermal elasticity equations for inhomogeneous plates [15]. Here $\lambda = \lambda(r, \varphi, z)$, $\mu = \mu(r, \varphi, z)$ are Lamé coefficients and $\alpha_t \equiv \alpha_t(r, \varphi, z)$ is the temperature coefficient of linear expansion of piecewise homogeneous plates. After substitution of the thermoelastic characteristics (3), using the filtering property of the delta function and carrying out direct integration reduce relations (6) to

$$J_1 = 2\delta[\gamma_1 + 2h(\gamma_0 - \gamma_1)\delta(r - R)N(\varphi)],\ J_2 = 2\delta[\mu_1 + 2h(\mu_0 - \mu_1)\delta(r -$$
$$- R)N(\varphi)],\ J_3 = 2\delta[\chi_1 + 2h(\chi_0 - \chi_1)\delta(r - R)N(\varphi)](T - t_0),\ J_4 = 0,$$

$$J_5 = \frac{2}{3}\delta^3[\gamma_1 + 2h(\gamma_0 - \gamma_1)\delta(r-R)N(\varphi)],\ J_1^* = 0,\quad J_2^* = 0,\quad J_3^* = 0, \tag{7}$$

$$J_4^* = \frac{2}{3}\delta^3[\Psi_1 + 2h(\Psi_0 - \Psi_1)\delta(r-R)N(\varphi)],$$

$$J_2^{(1)} = \frac{2}{3}\delta^3[\mu_1 + 2h(\mu_0 - \mu_1)\delta(r-R)N(\varphi)].$$

In the above expressions the definitions are as follows:

$$\gamma_i = \frac{\lambda_i\mu_i}{\lambda_i + 2\mu_i} = \frac{E_i\nu_i}{2(1-\nu_i^2)};\ \Psi_i = \frac{\mu_i(\lambda_i+\mu_i)}{\lambda_i + 2\mu_i} = \frac{E_i}{4(1-\nu_i^2)};$$

$$\chi_i = \frac{\mu_i\beta_i}{\lambda_i + 2\mu_i} = \frac{\alpha_t^{(i)}E_i}{2(1-\nu_i)};\ \beta_i = 3\lambda_i + 2\mu_i;$$

E_i are the elasticity moduli; ν_i are the Poisson coefficients, i = 0.1.

The substitution of (7) into the thermal elasticity equation for inhomogeneous plates [15] gives, after some transformations, the following system of simultaneous partly degenerate differential thermal elasticity equations in terms of displacements for a plate with a through annular sector-shaped inclusion located symmetrically relative to the median plane:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{2(\gamma_i+\mu_i)}\left[\frac{\mu_1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{2\gamma_1+\mu_1}{r}\frac{\partial^2 v}{\partial r\partial\varphi} - \right.$$
$$\left. - \frac{2\gamma_1 + 3\mu_1}{r^2}\frac{\partial v}{\partial\varphi} - 2\chi_1\frac{\partial(T-t_0)}{\partial r}\right] = -\frac{2h}{\gamma_1+\mu_1}\left\langle\left\{[\gamma_0 + \mu_0 - \right.\right.$$
$$\left. - (\gamma_1+\mu_1)]\frac{\partial u}{\partial r}\Big|_{r=R} + \frac{\gamma_0-\gamma_1}{R}\left(u + \frac{\partial v}{\partial\varphi}\right)_{r=R} - (\chi_0-\chi_1)(T-t_0)|_{r=R}\right\}\times$$
$$\times N(\varphi)\delta'(r-R) + \frac{\mu_0-\mu_1}{R}\left\{\left(\frac{\partial u}{\partial r} + \frac{1}{2R}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{2}\frac{\partial^2 v}{\partial r\partial\varphi} - \frac{3}{2R}\frac{\partial v}{\partial\varphi} - \right.\right.$$
$$\left. - \frac{u}{R}\right)_{r=R} N(\varphi) + \frac{1}{2R}\left[\left(\frac{\partial u}{\partial\varphi} - v + R\frac{\partial v}{\partial r}\right)_{\substack{r=R\\ \varphi=-\varphi_0+0}}\delta_-(\varphi+\varphi_0) - \right.$$
$$\left.\left.\left. - \left(\frac{\partial u}{\partial\varphi} - v + R\frac{\partial v}{\partial r}\right)_{\substack{r=R\\ \varphi=\varphi_0-0}}\delta_+(\varphi-\varphi_0)\right]\right\}\delta(r-R)\right\rangle.$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{\mu_1 r}\left[2(\gamma_1+\mu_1)\frac{1}{r}\frac{\partial^2 v}{\partial\varphi^2} + (2\gamma_1+\mu_1)\frac{\partial^2 u}{\partial r\partial\varphi} + \right. \tag{8}$$
$$\left. + \frac{2\gamma_1+3\mu_1}{r}\frac{\partial u}{\partial\varphi} - 2\chi_1\frac{\partial(T-t_0)}{\partial\varphi}\right] = -\frac{2h(\mu_0-\mu_1)}{\mu_1}\left(\frac{\partial v}{\partial r} - \frac{v}{R} + \right.$$
$$\left. + \frac{1}{R}\frac{\partial u}{\partial\varphi}\right)_{r=R} N(\varphi)\delta'(r-R) - \frac{4h}{\mu_1 R}\left\{(\mu_0-\mu_1)\left[\frac{\partial v}{\partial r} + \left(1 + \frac{\gamma_0-\gamma_1}{\mu_0-\mu_1}\right)\times\right.\right.$$

$$\left. \times \frac{\partial u}{\partial\varphi} + \frac{\gamma_0-\gamma_1}{\mu_0-\mu_1}\frac{\partial^2 u}{\partial r\partial\varphi} - \frac{\chi_0-\chi_1}{\mu_0-\mu_1}\frac{\partial(T-t_0)}{\partial\varphi}\right]_{r=R} N(\varphi) +$$
$$+ \left[\frac{\gamma_0-\gamma_1+\mu_0-\mu_1}{R}\left(\frac{\partial v}{\partial\varphi} + u\right) + (\gamma_0-\gamma_1)\frac{\partial u}{\partial r} - (\chi_0-\chi_1)(T - \right.$$
$$\left. - t_0)\right]_{\substack{r=R\\ \varphi=-\varphi_0+0}}\delta_-(\varphi-\varphi_0) - \left[\frac{\gamma_0-\gamma_1+\mu_0-\mu_1}{R}\left(\frac{\partial v}{\partial\varphi} + u\right) + \right.$$
$$\left.\left. + (\gamma_0-\gamma_1)\frac{\partial u}{\partial r} - (\chi_0-\chi_1)(T-t_0)\right|_{\substack{r=R\\ \varphi=\varphi_0-0}}\delta_+(\varphi-\varphi_0)\right\}\delta(r-R).$$

Let us consider the case where the plate contains a through foreign wedge-shaped inclusion (Fig. 2). Then, the physical and mechanical properties are functions of the coordinate φ alone. Now, the heat-conduction and thermal elasticity equations will be written for this case. Then, relations (4) will become

$$\Lambda(\varphi) = 2\delta[\lambda_t^{(1)} + (\lambda_t^{(0)} - \lambda_t^{(1)})N(\varphi)],\ \Lambda^*(\varphi) = 0,$$
$$C(\varphi) = 2\delta[c_v^{(1)} + (c_v^{(0)} - c_v^{(1)})N(\varphi)],\ C^*(\varphi) = 0, \tag{9}$$

and relations (6) will be written as follows:

$$J_1 = 2\delta[\gamma_1 + (\gamma_0 - \gamma_1)N(\varphi)],\ J_2 = [2\delta[\mu_1 + (\mu_0 - \mu_1)N(\varphi)],$$
$$J_3 = 2\delta[\chi_1 + (\chi_0 - \chi_1)N(\varphi)](T - t_0),\ J_4 = 0,\ J_5 = \frac{2}{3}\delta^3[\gamma_1 + (\gamma_0 -$$
$$- \gamma_1)N(\varphi)],\ J_1^* = 0,\ J_2^* = 0,\ J_3^* = 0,\ J_4^* = \frac{2}{3}\delta^3[\Psi_1 + (\Psi_0 - \Psi_1)N(\varphi)], \tag{10}$$
$$J_2^{(1)} = \frac{2}{3}\delta^3[\mu_1 + (\mu_0 - \mu_1)N(\varphi)].$$

Substituting (9) into (1) and (10) into the thermal elasticity equation for inhomogeneous plates [15] and taking the structure of multiplying asymmetric unit functions [17] into account, we arrive at the following heat conduction equation and simultaneous system of thermal elasticity equations for a plate containing a wedge-shaped inclusion:

$$\Delta T - \varkappa_1^2(T - t_c) = \frac{1}{a_1}\dot{T} + (1 - K_\lambda)\frac{1}{r^2}\left[\frac{\partial T}{\partial\varphi}\bigg|_{\varphi=-\varphi_0+0}\delta_-(\varphi + \varphi_0) -\right.$$
$$\left. - \frac{\partial T}{\partial\varphi}\bigg|_{\varphi=\varphi_0-0}\delta_+(\varphi - \varphi_0)\right] - \left[\left(\frac{1}{a_1} - \frac{1}{a_0}\right)\dot{T} + (\varkappa_1^2 - \varkappa_0^2)(T - t_c)\right]N(\varphi); \tag{11}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{2(\varphi_1 + \mu_1)}\left[\frac{\mu_1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{2\gamma_1 + \mu_1}{r}\frac{\partial^2 v}{\partial r\partial\varphi} -\right.$$
$$\left. - \frac{1}{r^2}(2\gamma_1 + 3\mu_1)\frac{\partial v}{\partial\varphi} - 2\chi_1\frac{\partial(T - t_0)}{\partial r}\right] = -\frac{1}{2r^2}\left\{\left[\left(\frac{\mu_0}{\gamma_0 + \mu_0} -\right.\right.\right.$$
$$\left. - \frac{\mu_1}{\gamma_1 + \mu_1}\right)\frac{\partial^2 u}{\partial\varphi^2} + \left(\frac{2\gamma_0 + \mu_0}{\gamma_0 + \mu_0} - \frac{2\gamma_1 + \mu_1}{\gamma_1 + \mu_1}\right)r\frac{\partial^2 v}{\partial r\partial\varphi} - \left(\frac{2\gamma_0 + 3\mu_0}{\gamma_0 + \mu_0} -\right.$$
$$\left. - \frac{2\gamma_1 + 3\mu_1}{\gamma_1 + \mu_1}\right)\frac{\partial v}{\partial\varphi} - 2\left(\frac{\chi_0}{\gamma_0 + \mu_0} - \frac{\chi_1}{\gamma_1 + \mu_1}\right)r^2\frac{\partial(T - t_0)}{\partial r}\right]N(\varphi) +$$
$$+ \frac{\mu_0 - \mu_1}{\gamma_1 + \mu_1}\left[\left(\frac{\partial u}{\partial\varphi} + r\frac{\partial v}{\partial r} - v\right)_{\varphi=-\varphi_0+0}\delta_-(\varphi + \varphi_0) - \left(\frac{\partial u}{\partial\varphi} + r\frac{\partial v}{\partial r} -\right.\right.$$
$$\left.\left.\left. - v\right)_{\varphi=\varphi_0-0}\delta_+(\varphi - \varphi_0)\right]\right\},$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{1}{\mu_1 r}\left[2(\gamma_1 + \mu_1)\frac{1}{r}\frac{\partial^2 v}{\partial\varphi^2} + (2\gamma_1 + \mu_1)\frac{\partial^2 u}{\partial r\partial\varphi} +\right.$$
$$\left. + \frac{1}{r}(2\gamma_1 + 3\mu_1)\frac{\partial u}{\partial\varphi} - 2\chi_1\frac{\partial(T - t_0)}{\partial\varphi}\right] = -\frac{1}{r^2}\left[\left(\frac{2\gamma_0 + \mu_0}{\mu_0} -\right.\right.$$
$$\left. - \frac{2\gamma_1 + \mu_1}{\mu_1}\right)r\frac{\partial^2 u}{\partial r\partial\varphi} + \left(\frac{2\gamma_0 + 3\mu_0}{\mu_0} - \frac{2\gamma_1 + 3\mu_1}{\mu_1}\right)\frac{\partial u}{\partial\varphi} + 2\left(\frac{\gamma_0 + \mu_0}{\mu_0} -\right.$$
$$\left.\left. - \frac{\gamma_1 + \mu_1}{\mu_1}\right)\frac{\partial^2 v}{\partial\varphi^2} - 2\left(\frac{\chi_0}{\mu_0} - \frac{\chi_1}{\mu_1}\right)r\frac{\partial(T - t_0)}{\partial\varphi}\right]N(\varphi) - \frac{2}{\mu_1 r^2}\left\{[\gamma_0 +\right. \tag{12}$$
$$+ \mu_0 - (\gamma_1 + \mu_1)]\left[\left(\frac{\partial v}{\partial\varphi} + u\right)_{\varphi=-\varphi_0+0}\delta_-(\varphi + \varphi_0) - \left(\frac{\partial v}{\partial\varphi} + u\right)_{\varphi=\varphi_0-0}\delta_+(\varphi -\right.$$
$$\left. - \varphi_0)\right] + (\gamma_0 - \gamma_1)r\left[\frac{\partial u}{\partial r}\bigg|_{\varphi=-\varphi_0+0}\delta_-(\varphi + \varphi_0) - \frac{\partial u}{\partial r}\bigg|_{\varphi=\varphi_0-0}\delta_+(\varphi - \varphi_0)\right] -$$
$$\left. - (\chi_0 - \chi_1)r[(T - t_0)_{\varphi=-\varphi_0+0}\delta_-(\varphi + \varphi_0) - (T - t_0)|_{\varphi=\varphi_0-0}\delta_+(\varphi - \varphi_0)]\right\}.$$

Equations (11) and (12) determine the thermally stressed state of the operating element of a MCC with a foreign wedge-shaped inclusion.

The derivation of the reduced partly degenerate differential equations will be illustrated by an example of solving the heat conduction problem for a thin plate with a wedge-shaped inclusion, the apex of which lies at the coordinate origin. The angular opening of the inclusion is $2\varphi_0$, the plate is assumed to be thermally insulated over the surfaces $\varphi = 0, \pi$ and $z = \pm\delta$ and at a distance R from the coordinate origin it is assumed to be heated by a fixed heat source with a variable specific heat flux q_i. In this case the source is a hollow thin-walled cylinder, 2δ in height, with the wall thickness 2h. This case can be considered in determining the temperature gradients in inhomogeneous MCC elements during brazing of the leads to the elements, which, in turn, gives rise to temperature stresses bringing about various kinds of defects in the ceramics.

Using Eq. (11) and taking account of the effect of the cylindrical heat source, after transformations we obtain the following partly degenerate differential heat conduction equation for determination of the two-dimensional quasisteady temperature field in a plate containing a wedge-shaped foreign inclusion [18]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} = (1 - K_\lambda)\frac{1}{r^2}\left[\frac{\partial T}{\partial \varphi}\bigg|_{\varphi=-\varphi_0+0} \delta_-(\varphi + \varphi_0) - \right.$$
$$\left. - \frac{\partial T}{\partial \varphi}\bigg|_{\varphi=\varphi_0-0} \delta_+(\varphi - \varphi_0)\right] - Q_1[1 - (1 - K_Q)N(\varphi)]\frac{\delta(r-R)}{R} \tag{13}$$

and the boundary conditions

$$\frac{\partial T}{\partial \varphi}\bigg|_{\substack{\varphi=0\\ \varphi=\pi}} = 0, \tag{14}$$

where $K_Q = Q_0/Q_1$; $Q_i = q/4\pi\lambda^{(i)}{}_t\delta$; q_0 and q_1 are the specific heat fluxes from the heat sources in the inclusion and in the matrix, respectively.

Applying the Mellin integral transform to the boundary-value problem (13), (14) for the variable r [19]

$$\overline{T}(n, \varphi) = \int_0^\infty T(r, \varphi)\, r^{n-1}\, dr \tag{15}$$

($n = \sigma + i\infty$ is the complex variable in the Mellin transform) and using the properties of the delta function [17], we arrive at the heat-conduction equation for a system in the Mellin transform space

$$\frac{d^2\overline{T}}{d\varphi^2} + n^2\overline{T} = (1 - K_\lambda)\left[\frac{d\overline{T}}{d\varphi}\bigg|_{\varphi=-\varphi_0+0} \delta_-(\varphi + \varphi_0) - \right.$$
$$\left. - \frac{d\overline{T}}{d\varphi}\bigg|_{\varphi=\varphi_0-0} \delta_+(\varphi - \varphi_0)\right] - Q_1 R^n [1 - (1 - K_Q)N(\varphi)], \tag{16}$$

$$\frac{d\overline{T}}{d\varphi}\bigg|_{\substack{\varphi=0\\ \varphi=\pi}} = 0. \tag{17}$$

The general solution of the transformed heat-conduction equation (16) will be expressed as

$$\overline{T} = A\cos n\varphi + B\sin n\varphi - \frac{1}{2ni}(1 - K_\lambda)\left(\frac{d\overline{T}}{d\varphi}\bigg|_{\varphi=-\varphi_0+0} \exp(-ni|\varphi + \right.$$
$$\left. + \varphi_0|_-) - \frac{d\overline{T}}{d\varphi}\bigg|_{\varphi=\varphi_0-0} \exp(-ni|\varphi - \varphi_0|_+)\right) - \frac{Q_1 R^n}{n^2}\left\{1 + \frac{1}{2} \times \right.$$
$$\times (1 - K_Q)[\mathrm{sign}_-(\varphi + \varphi_0)\exp(-ni|\varphi + \varphi_0|_-) - \mathrm{sign}_+(\varphi - \varphi_0) \times$$
$$\times \exp(-ni|\varphi - \varphi_0|_+) - 2N(\varphi)],$$

where $|\varphi \pm \varphi_0|_\mp = (\varphi \pm \varphi_0)\,\mathrm{sign}_\mp(\varphi \pm \varphi_0)$, $\mathrm{sign}_\mp(\varphi \pm \varphi_0) = 2S_\mp(\varphi \pm \varphi_0) - 1$, $i = \sqrt{-1}$.

In order to determine the boundary values of the derivative of the temperature transform $d\bar{T}/d\varphi\,|_{\varphi=\varphi\pm 0}$ and the integration constants A and B, appropriate sets of algebraic equations are solved on the basis of (17) and (18). With this in view, the final analytical expression for the Mellin transform of the temperature field in a plate with a wedge-shaped inclusion will be written as

$$\bar{T}=\frac{Q_1R^n}{2n^2i}(K_Q-1)\left\{\frac{1-K_\lambda}{1+K_\lambda}(1-\exp(-2ni\varphi_0))[\exp(-ni(\pi+\varphi_0))+\text{sign}_+(\pi-\right.$$
$$-\varphi_0)\exp(-ni(\pi-\varphi_0))]+\left(1+\frac{1-K_\lambda}{1+K_\lambda}\exp(-2ni\varphi_0)\right)(\exp(-ni(\pi+$$
$$\left.+\varphi_0))-\exp(-ni(\pi-\varphi_0)))\right\}\left[\cos n\varphi-\frac{1-K_\lambda}{1+K_\lambda}\frac{\sin n\varphi_0}{i}(\exp(-ni|\varphi+\right.$$
$$\left.+\varphi_0|_-)+\exp(-ni|\varphi-\varphi_0|_+))\left(1+\frac{1-K_\lambda}{1+K_\lambda}\exp(-2ni\varphi_0)\right)^{-1}\right]\times$$
$$\times\left[\sin n\pi\left(1+\frac{1-K_\lambda}{1+K_\lambda}\exp(-2ni\varphi_0)\right)-\frac{1-K_\lambda}{1+K_\lambda}\sin n\varphi_0(\exp(-ni(\pi+\right. \qquad (19)$$
$$\left.+\varphi_0))+\text{sign}_+(\pi-\varphi_0)\exp(-ni(\pi-\varphi_0))\right]^{-1}+\left[\frac{1-K_\lambda}{1+K_\lambda}\frac{Q_1R^n}{2n^2}\times\right.$$
$$\times(K_Q-1)(1-\exp(-2ni\varphi_0))(\exp(-ni|\varphi+\varphi_0|_-)+\exp(-ni|\varphi-$$
$$\left.-\varphi_0|_+))\right]\left(1+\frac{1-K_\lambda}{1+K_\lambda}\exp(-2ni\varphi_0)\right)^{-1}-\frac{Q_1R^n}{n^2}\left\{1+\frac{1-K_Q}{2}\times\right.$$
$$\left.\times[\text{sign}_-(\varphi+\varphi_0)\exp(-ni|\varphi+\varphi_0|_-)-\text{sign}_+(\varphi-\varphi_0)\exp(-ni|\varphi-\varphi_0|_+)-2N(\varphi)]\right\}.$$

Inversion of the transform is performed using the Mellin inversion formula [19]

$$T(r,\varphi)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{T}(n,\varphi)r^{-n}dn. \qquad (20)$$

We will consider the limiting case of the formula (19) encountered in practice when the opening angle of a foreign wedge-shaped inclusion φ_0 is equal to $\pi/2$ (Fig. 3). Then, with the use of expression (20), the temperature field arising at the junction $\varphi = \varphi_0 = \pi/2$ of the two components of the semiinfinite heterogeneous plate (the bioplatelet case) will be found as

$$T(r)=\frac{Q_1(1-K_Q)(1-\beta^2)}{2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{R}{r}\right)^n\frac{\exp(-\pi in)\,dn}{n^2(1+\beta\exp(-\pi in))}-$$
$$-\frac{Q_1}{2}(1-K_Q)\beta\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{R}{r}\right)^n\frac{1-\exp(-2\pi in)}{n^2(1+\beta\exp(-\pi in))}dn- \qquad (21)$$
$$-Q_1\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{R}{r}\right)^n\frac{dn}{n^2}+\frac{Q_1}{2}(1-K_Q)\frac{1}{2\pi i}\times$$
$$\times\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{R}{r}\right)^n\frac{1}{n^2}(1-\exp(-\pi in))\,dn,$$

where $\beta = (1 - K_\lambda)/(1 + K_\lambda)$.

Finally, we will consider a case important for solution of the problem of heat removal in operating homogeneous ceramic substrates which are heated by a cylindrical heat source with radius R. Then the thermal properties of the inclusion material and the basic material of the system are practically the same. With $K_\lambda = K_Q = 1$, $Q_1 = Q = q/(4\pi\lambda_t\delta)$, $\varphi_0 = 0$ assumed in the heat conduction equation (13), it will be written in a form given earlier in the literature [20]:

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}=-Q\frac{\delta(r-R)}{R}. \qquad (22)$$

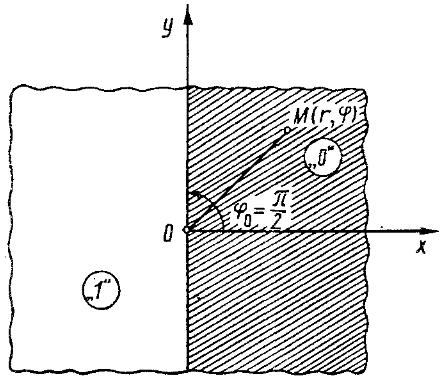

Fig. 3. A bioplatelet (the limiting case of a plate with a foreign inclusion shaped like a wedge with the opening angle $\varphi_0 = \pi/2$ (rad)).

A general solution of the above equation is easily obtained by direct integration:

$$T(r) = Q \ln \frac{R}{r} S(r-R) + C \ln \frac{r}{R} + D, \tag{23}$$

where C and D are integration constants determined from the respective boundary conditions; $S(r - R) = \begin{cases} 1, & r > R, \\ 0{,}5, & r = R, \\ 0, & r < R. \end{cases}$

Thus, heat conduction equations (5) and (11) and thermal elasticity equations (8) and (12) as well as analytical expressions (19), (20) and (23) are used in the description of thermal processes occurring in inhomogeneous operating MCC elements with inclusions of the type considered in specific cases of their heating during their manufacture and operation. Then, thermal fields are necessary for the analysis of thermal stresses which produce flaws in metal ceramics such as microcracks, warping, lamination, bloating of inhomogeneous components in operating units, rupture of inner conductors. Moreover, the method suggested for the derivation of differential heat conduction and thermal elasticity equations can be effectively used for determining thermally stressed states of inhomogeneous components with various kinds of through and nonthrough foreign inclusions which must be obligatorily taken into consideration while designing MCC. However, nonthrough inclusions make the problem much more complicated as its dimension increases thereby [17]. For example, for a plate with through foreign inclusions the one-dimensional heat conduction problem becomes two-dimensional in the case of a nonthrough inclusion. The method suggested for finding the solution is also effective in that case, but it should be considered separately.

NOTATION

T_{melt}, melting temperature of the material (oxide ceramics); TCLE $\equiv \alpha_t$, temperature coefficient of linear expansion; 2δ, thickness of a thin plate; R, 2h, median radius and wall thickness of the foreign inclusion; t^+_f, t^-_f, temperatures of fluids flowing over the upper $z = +\delta$ and lower $z = -\delta$ plate surfaces; r, φ, z, components of the three-dimensional rectangular coordinate system (r, φ, z); Λ, Λ^*, Λ^{**}, coefficients in the form of definite integrals of thermal conductivities entering into the system of simultaneous differential heat conduction equations; T, T^*, integral characteristics of the temperature field of an inhomogeneous plate; τ, time; Δ, Laplace operator; α^t_z, coefficients of heat transfer from the plate surfaces $z = \pm\delta$; $t^f_\pm$, half-sum and half-difference of temperatures for fluids flowing over the plate surfaces $z = \pm\delta$; C, C^*, C^{**}, coefficients in the form of definite integrals of the volumetric heat capacity coefficients entering into the system of simultaneous heat conduction equations; T, T^*, first derivatives of integral characteristics of the system in time τ; t^*, temperature increment; t_0, initial temperature at which there are no stresses in the body; $c_v(r, \varphi, z)$, $c(r, \varphi, z)$, volumetric and specific heat capacities of an inhomogeneous plate; $\rho(r, \varphi, z)$, density of the system material; p_1, p_0, physical and mechanical properties (any) of the basic material (matrix) and the material of an inclusion in the plate and their combinations; $\delta(r - R)$, $N(\varphi)$, generalized functions, delta function in the radius, and characteristic functions in the angle; $S_\pm(r - R \pm h)$, asymmetric generalized unit functions; λ^0_t, λ^1_t, thermal conductivities of the inclusion material and the basic material of the system; $\varkappa_0$, $\varkappa_1$, reduced coefficients of heat transfer from the surfaces of an inclusion and the matrix; a_1, thermal diffusivity of the basic material; $\delta'(r - R)$, first derivative of the delta function in the radial coordinate; K_λ, criterion characterizing the relative thermal conductivity of the components of a piecewise homogeneous body; $\delta_\pm(\varphi \pm \varphi_i)$, asymmetric delta functions defined by $\delta_\pm(\varphi \pm \varphi_0) = dS \pm (\varphi \pm \varphi_0)/d\varphi$; $J_1, J_2, \ldots, J_5$; $J^*_1, \ldots, J^*_4$; $J^{(1)}_2$, coefficients in the form of definite integrals of Lamé coefficients in the simultaneous differential thermal elasticity equations; $\lambda(r, \varphi, z)$, $\mu(r, \varphi, z)$, Lamé coefficients of a piecewise homogeneous body; $\alpha_t(r, \varphi, z)$, TCLE of a piecewise homogeneous body; γ_i, $\varkappa_i$, Ψ_i ($i = 0, 1$), combination of Lamé coefficients and the TCLE of a piecewise homogeneous body; E_i, ν_i ($i = 0, 1$), elasticity moduli and Poisson coefficients for the components of a piecewise homogeneous body; β_i, coefficients in the thermal elasticity equations characterizing temperature changes of the components of a piecewise homogeneous body as a function of the TCLE; u, v, w, displacement vector components for an inhomogeneous body (in the thermal elasticity problem considered for a MCC $w = 0$); q_i, specific heat fluxes from the heat source active in a piecewise homogeneous body; K_Q, criterion characterizing the thermal flux of the components of a piecewise homogeneous system; Q_i, reduced specific heat fluxes of heat sources in each component separately; n, complex variable of the Mellin integral transform ($n = \sigma + i\infty$); T, desired function in the Mellin transform space; $|\varphi - \varphi_0|$, asymmetric generalized modulus functions for the coordinate of the angle; $\mathrm{sign}_\pm(\varphi \pm \varphi_0)$, asymmetric generalized sign functions for the coordinate of the angle; $i = \sqrt{(-1)}$, imaginary unit; β, coefficient characterizing the relative excess of thermal conductivity K_λ; $S(r - R)$, total (symmetrical) generalized unit function.

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